The Bicyclist's Paradox

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It's a situation every avid cyclist knows only too well. If you cycle up a hill and then back down with no net change in elevation, it seems as if your slower uphill speed and faster downhill speed should offset each other. But they don't. Your average speed is less than it would have been had you cycled the same distance on a level road. Similarly, cycling into a headwind for half your trip and returning home with a tailwind yields an average speed less than you would have achieved on a windless day. The faster part of the ride doesn't compensate for the slower part. It seems unjust. Most cyclists expect the uphill and downhill, or the headwind and tailwind, to more or less cancel and are surprised (and frustrated!) when they don't. The purpose of this paper is to resolve this paradox. Doing so involves some nice real-world applications of Newton's laws, numerical problem solving, and exercise physiology. There's a lot to learn from analyzing this problem, and it's readily accessible to introductory physics students.

Consider first a bicycle traveling on level ground. Three forces act parallel to the direction of motion: A propulsion force F_{prop} in the forward direction, a retarding friction force F_{fric}, and a wind-resistance force F_{wind}. The propulsion force F_{prop} is really $F_{road\ on\ tire}$, the third-law reaction to the force $F_{tire\ on\ road}$ that the cyclist exerts on the road via a gear-and-chain system when he presses down on one pedal. You push against the road, and it pushes against you, driving you forward. Propulsion forces are difficult for beginning students to understand, and they require the system to have an internal source of energy—gasoline for your car, granola bars for the cyclist. We'll return to the energy aspects later.

There are other friction forces—in the chain and the wheel bearings—but these are internal forces. They affect how much of the cyclist's expenditure of energy is ultimately used to propel the bike, but only these three external forces appear in Newton's second law for the bicycle.

According to Newton's laws, there is no net force on a bicycle moving in a straight line at constant speed. Thus the propulsion force must exactly balance the friction and wind-resistance forces:

$$F_{prop} = F_{wind} + F_{fric}\,. \qquad (1)$$

The power required to propel the bike forward at speed v is $P = F_{prop}v$, and thus

$$P = (F_{wind} + F_{fric})v\,. \qquad (2)$$

This is the power the cyclist must generate to ride at a steady speed v.

The friction force in Eq. (2) is rolling friction. It can be modeled analogously to sliding or kinetic friction: $F_{fric} = \mu_r n$, where μ_r is the coefficient of rolling friction and n is the normal force due to the road surface. The normal force is simply mg for an object on a horizontal surface, so a reasonable model of the friction force is $F_{fric} = \mu_r mg$.

Wind resistance is trickier. Many textbooks give the drag force as $\frac{1}{2}C_D\rho Av^2$, where ρ is the density of air and A is the object's cross section perpendicular to

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the direction of motion. C_D is the "drag coefficient," a number that depends sensitively on the shape and smoothness of the object. Although the drag coefficient depends weakly on v, we'll consider C_D to be a constant that, like μ_r, must be determined empirically.

The remaining complication is that a bicycle is usually not moving through still air. The v is actually the *relative* speed between the bicycle and the air. If we define a tailwind (wind blowing in the direction of the bicycle's motion) as a positive wind velocity, and thus a headwind as negative, then the cyclist's *velocity* (not speed, because it could be negative) relative to the wind is $v_{rel} = v - v_{wind}$.

Now we have to be careful. If we simply square v_{rel}, then F_{wind} will always be a positive number and—as we've defined it—always represent a retarding force. But the wind *pushes* the bicycle if $v_{wind} > v$, for which F_{wind} needs to be a negative number. Although it looks rather complex, we can deal with an object moving through wind and get the signs right by writing

$$F_{wind} = \tfrac{1}{2} C_D \rho A (v - v_{wind}) |v - v_{wind}|. \quad (3)$$

Returning to Eq. (2), we can now write the power required to propel the bicycle at speed v as

$$\begin{aligned} P &= \tfrac{1}{2} C_D \rho A (v - v_{wind}) |v - v_{wind}| v + \mu_r m g v \\ &= a (v - v_{wind}) |v - v_{wind}| v + b v, \end{aligned} \quad (4)$$

where $a = \frac{1}{2} C_D \rho A$ and $b = \mu_r mg$ are constants for a particular bicyclist and bicycle.

Let's assume that the cyclist rides with a steady power output. This is a reasonable assumption for a racing cyclist and for a "serious rider" who rides for exercise. He or she shifts gears as needed to maintain a fairly steady heart rate and breathing rate—that is, a fairly steady power output. A cyclist can calculate his steady power output, once a and b have been determined, by measuring his cruising speed, which we'll call v_0, on a windless day: $P = av_0^3 + bv_0$. Cycling with this same power output, his speed (on level ground) on a day when the wind velocity is v_{wind} is the solution to the equation

$$a (v - v_{wind}) |v - v_{wind}| v + bv - P = 0. \quad (5)$$

Similar models of cycling, but without the effect of wind, have been considered by others.[1,2]

A Concrete Example

I'll use my own riding experience to make these ideas concrete, but an excellent lesson is to let students do the calculations with numbers appropriate to them. I ride a lightweight road bike (dropped handlebars) primarily for aerobic exercise, so I push myself reasonably hard and, as judged by heart and breath rate, I do ride at a fairly steady power output. I live in the coastal mountains of California, about a mile from the ocean, so neither windless days nor long, level stretches of road are common occurrences. Nonetheless, riding along short, level stretches on near-windless days suggests that I could maintain a 20 mph pace for one hour (a typical ride duration) should I ever find that long, level, windless highway. This is my value of v_0.

To determine my personal values of a and b, we turn to the "bible" of factual information about cycling, *Bicycling Science* by David Wilson.[3] (Earlier editions were by Frank Whitt and David Wilson). It is a book both fascinating and frustrating. The data tables and graphs are taken from a wide variety of sources, and in many cases the reader doesn't know the conditions under which measurements were made. Thus reported results on, say, the coefficient of rolling friction for bike tires span a rather wide range, and trying to decide what value is most appropriate to you has to be somewhat of an educated guess.

Coefficients of rolling friction are reported from as low as 0.002 for racing tires to 0.015 for "wide tires on rough surfaces," with "wide tires" undefined. The lowest value appears to be measured on a machine where the tire turns against hard, smooth rollers. That might be relevant to indoor racing on a wooden track, but rolling friction is certainly larger on asphalt pavement. After scrutinizing various graphs and tables, I settled on $\mu_r = 0.006$ as a plausible value. The bike and I have a combined mass of 85 kg, leading to $b = \mu_r mg =$ 5.0 N.

The one well-known value for determining a is the density of air at 20°C, $\rho = 1.2$ kg/m^3. My cross-section area is not easily calculated, but let's say my average width is 15 in = 0.38 m and that my "height" from knees to head while in a biking position is 42 in =

1.07 m. My calves and feet present much less area and are often bent out of the wind while pedaling, so I'll ignore those. Thus my estimated area is A = 0.38 m × 1.07 m = 0.41 m^2. As for the drag coefficient C_D, I'll adopt the value 1.0 Wilson gives for a road bike with the cyclist in a "touring position" with hands on top the brakes. (Sitting upright increases C_D about 10% whereas a full racer's crouch lowers it about 10%.) I ride in somewhat of a crouch but probably not enough to change C_D. With these choices, I find

$$a = \frac{1}{2} C_D \rho A = 0.25 \text{ kg/m}.$$

I can now calculate that my personal power level is

$$P = a v_0^3 + b v_0 = (0.25 \text{ kg/m})(8.9 \text{ m/s})^3 + (5.0 \text{ N})(8.9 \text{ m/s}) = 176 \text{ W} + 44 \text{ W} = 220 \text{ W}, \quad (6)$$

where I converted v_0 = 20 mph to 8.9 m/s. Note that this is the power used to propel the bike; my actual power output must be slightly higher because of friction in the drive train. Wind resistance is far more important than rolling friction at this speed.

The assumption is that I'll continue to provide 220 W even when the wind blows, adjusting my speed to maintain the same level of effort. According to Eq. (5), my speed when the wind velocity is v_{wind} is the solution to

$$0.25(v - v_{\text{wind}})\left|v - v_{\text{wind}}\right|v + 5.0v - 220 = 0. \quad (7)$$

I've suppressed units to keep the equation straightforward, but it is *very important* when doing calculations to remember that v and v_{wind} must be in m/s.

Students—and most instructors!—do not know how to solve cubic equations in closed form. Fortunately, there's no need to. It is easy to solve equations such as this in Excel by using the procedure called Solver, and doing so is an excellent opportunity for students to gain early experience with numerical problem solving. Type an "initial guess" for v into a cell in column A; your personal value for v_0 is a very reasonable initial guess. Then enter the left-hand side of Eq. (7) into the adjacent cell in column B, starting with = to make it an Excel equation. Refer to the column A cell for the value of v and to some nearby cell where you'll enter a value of v_{wind}. The value calculated for the column B cell will not be 0 *unless* the value of v in column A is the solution to Eq. (7). You could imagine—and it's worth having students try this once—manually adjusting the value of v in column A, using trial and error, until you find a value that makes the column B cell zero. You've then "solved" the equation.

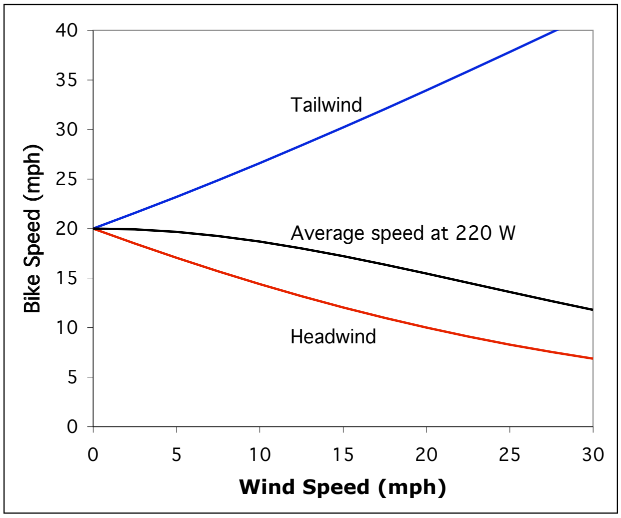

Fig. 1. The effect of riding with the wind or against the wind at a steady power output. The average speed over an out-and-back route decreases as the wind speed increases. These graphs are based on my riding characteristics; graphs for other riders will have similar shapes but different numerical values.

That's what Solver does automatically. Select Solver from the Tools menu, give it the column B cell address as the Target Cell, and tell it you want the Target Cell to have the value of 0. Give it the column A cell address as the cell whose value you want to vary to solve the equation, then click Solve. For the graphs shown, I varied the wind speed or slope angle through a predetermined range of values, used Solver to find the bike speed at each value, and saved each result into another column of the spreadsheet from which I could convert them to mph and make a chart.

The top and bottom curves in Fig. 1 are my speed riding with the wind and heading into the wind, all at my personal power level of 220 W. The middle curve—the basis for the bicycler's paradox—is my average speed over an out-and-back route, assuming the wind parallels my course rather than being a cross wind. The values in this graph are in good agreement with my personal experience.

With an out-and-back route, I have a headwind and a tailwind for equal distances d. Average speed is total distance $2d$ divided by total time. If my speeds out and back are v_{out} and v_{back}, my times out and back are, respectively, d/v_{out} and d/v_{back}. Thus my average speed is

$$v_{avg} = \frac{2}{1/v_{out} + 1/v_{back}}. \quad (8)$$

As the middle curve shows, my average speed drops steadily as the wind speed increases; the faster downwind speed does not compensate for the slower upwind speed.

If you drive 10 mph and 30 mph for equal *times*, your average speed is the average of the two speeds, namely 20 mph. A faster speed really can compensate for a slower speed over equal time intervals. But over equal *distances*, the reciprocal of v_{avg} is the average of the reciprocals of the two speeds. When taking reciprocals, a smaller speed has disproportionate influence. If the slower speed drops below $\frac{1}{2}v_0$—which for me happens at a headwind speed of ≈20 mph—then $v_{avg} < v_0$ regardless of how large the faster speed is.

What else can we learn from Eq. (7)? For example, at what tailwind speed would I "ride with the wind," having a relative wind speed of zero and thus experiencing no wind resistance? Setting the first term in Eq. (7) equal to zero, it's easy to find v = 44 m/s = 99 mph. I look forward to this day! Until then, as long as I pedal to maintain my steady 220 W, my speed over the ground is always larger than the tailwind speed and thus the relative wind is always in my face.

But suppose I stopped pedaling and let a tailwind push me? If the 220 W in Eq. (7) is replaced with 0, it's easy to find, after converting m/s to mph, that $v = v_{wind} - 10$ mph with my values of a and b. With a 15 mph tailwind I could cruise at 30 mph by continuing to pedal at full power, or I could be pushed along at a gentle 5 mph with no effort at all.

Ups and Downs

Now consider going up and down a hill of angle θ on a day with no wind. A new force enters Newton's second law, namely the component of the gravitational force parallel to the ground: $F_{grav} = \pm\, mg \sin\theta$. We should also use $n = mg\cos\theta$ for the normal force; however, the $\cos\theta$ term makes no noticeable contribution until θ exceeds 10°, hills only professionals ride, so in the interest of simplicity I'll leave the rolling resistance as simply $F_{fric} = \mu_r mg = b$.

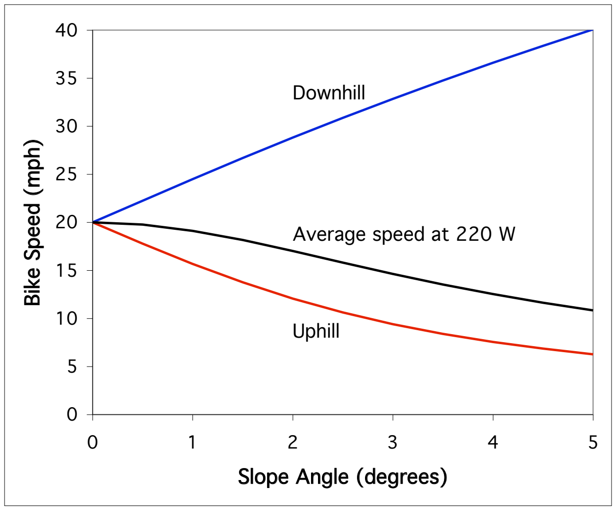

Fig. 2. The effect of riding up or down a hill at a steady power output, assuming no wind. The average speed over an up-and-down route decreases as the slope increases. These graphs are based on my riding characteristics; graphs for other riders will have similar shapes but different numerical values.

With a slope added, no wind, and mg = 833 N, Eq. (7) becomes

$$0.25v^3 + (5.0 \pm 833 \sin\theta)\, v - 200 = 0. \quad (9)$$

This is solved in Excel just as easily as was Eq. (7). Figure 2 shows the results for slopes up to 5°. A 5° slope may not sound like much, but it's an ascent of 460 ft/mile or, equivalently, an 8.7% grade. Highway slopes rarely exceed 3°. The uphill and downhill speeds shown in Fig. 2 agree reasonably well with my personal experience.

Just as with wind, the faster speed on a downhill does not compensate for the slower uphill speed. Consequently, the average speed of going up and down hills, with no net change in elevation, is less than could be achieved over level ground. In fact, the situation going up and down steeper hills (more than about 2.5°) is even worse than shown because few cyclists have the gearing or pedaling speed to maintain full power output at speeds in excess of 30 mph. The downhill ride may be exhilarating, but it's likely at a speed less than shown in Fig. 2, and thus the rider's average speed suffers even more.

Not surprisingly, adding wind to a hilly out-and-back ride is a double whammy, but this analysis will be left to the reader. My typical route ranges from level to slopes of about 3°, so I'll guess that an average slope for the entire ride is ≈1°. If I calculate an average speed for going up and down a 1° slope with a tailwind of 12 mph then returning against the wind, v_{ave} drops from the 18.2 mph shown in Fig. 1, for no hills, to 17.5 mph. And, indeed, that is just about my average speed for a one-hour ride on a typical day.

Exercise Physiology

I noted earlier that P is the power to propel the bike—220 W in my case. The rider's power output is larger, mostly due to friction in the chain. Various studies[3] have found that the efficiency of a clean, lubricated bike chain is ≈95%. Assuming that to be the case, my actual steady power output is 230 W, or 0.31 hp.

Is that plausible? Back in 1964, in the early days of the space program, NASA studied human power output by measuring the length of time for which volunteers could maintain a particular power output. "Healthy men" could sustain just over 200 W for one hour, while "first-class athletes" could achieve nearly 400 W for an hour.[3] World-class cyclists have been measured at 500 W for an hour. I'm no first-class athlete, and I'm probably not as young as the volunteers NASA tested, but 230 W for an hour's ride seems plausible for someone who rides enough to stay in shape.

A number of particular interest to anyone who exercises is how many calories he or she "burns" by doing so. The human body is ≈25% efficient at turning metabolic energy into mechanical energy, the other 75% being transferred to the environment as "waste heat" via respiration, perspiration, and radiation. Thus my metabolic power is

$$P_{metabolic} \approx 4 \times 230 \text{ W} = 920 \text{ W} = 920 \text{ J/s}. \quad (10)$$

Converting joules to calories and recalling that a food calorie is 1 Cal = 1000 cal, my metabolic power is 790 Cal/hour. So that one-hour ride burns approximately 800 Cal—surely enough to justify a piece of cheesecake for dessert.

References

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